Northwestern University Department of Physics and Astronomy

PHYSICS 428-1 QUANTUM FIELD THEORY I

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Course Webpage: `http://www.hep.anl.gov/ian/teaching/QFT/QFT_Fall08.html`

ASSIGNMENT #2

Due at 3:30 PM, October 9th

(Two pages and three problems.)

Reading Assignments:
(a) Read the handout by Professor Michael Dine at UC Santa Cruz on relativity at `http://scipp.ucsc.edu/~dine/ph217/217relativity.pdf`
(b) Section 3.1 of Peskin and Schroeder.

Problem 1
Do exercises a, e, and h in the reading assignment (a).

Problem 2
Do Problem 3.1 in Peskin and Schroeder. (We haven't introduced a Dirac spinor yet. However, you have all learned a two-component Pauli spinor in Quantum Mechanics. For the purpose of this problem, think of the two-component Weyl spinor as the Pauli spinor, which should enable you to do the problem.)

Problem 3
We would like to consider a two-dimensional quantum harmonic oscillator with the following Hamiltonian in Cartesian coordinate:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

where I have set $m = \omega = 1$ for simplicity. The operators satisfy $[x_i, p_j] = -i\delta_{ij}$
(a) Write down the Hamiltonian in terms of creation and annihilation operators $\{a_1^\dagger, a_2^\dagger, a_1, a_2\}$, which satisfy the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$. (All other commutators are zero.)
(b) The Hamiltonian obviously has rotational invariance in the two-dimensional space: $U(R)x_iU(R)^\dagger = D(R)_{ij}x_j$, where $D(R)$ is a 2×2 orthogonal matrix. A less obvious invariance is a complex rotation S in (a_1, a_2),

$$U(S)\, a_i\, U(S)^\dagger = D(S)_{ij} a_j,$$

where $D(S)$ is a complex 2×2 matrix. Work out the condition on $D(S)$ in order for the Hamiltonian to be invariant under S : $U(S)\, H\, U(S)^\dagger = H$. Show that different states related by an S transformation $|a\rangle = U(S)|b\rangle$ are degenerate in energy.

(c) Define the one-particle states $\{|i\rangle = a_i^\dagger|0\rangle, i = 1, 2\}$. We discussed in class that any operator can be expressed in terms of creation and annihilation operators. Consider a set of operators $\{T^a, a = 1, 2, 3\}$ whose effects on the one-particle states are

$$T^a|0\rangle = 0, \qquad T^a|i\rangle = \frac{1}{2}|j\rangle[\sigma^a]_{ij}$$

where σ^a's are the Pauli matrices and $[\sigma^a]_{ij}$ are the matrix elements of Pauli matrices. Find the representation of T^a in terms of the creation and annihilation operators.
(d) Use your result in (c) to compute the commutators $[T^a, a_i^\dagger]$.